Semidefinite Relaxations of Chance Constrained Algebraic Problems

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Let

- $q \in \mathbb{R}^m$ be a random variable with probability measure $\mu_q$
- $x \in \mathbb{R}^n$ be a decision variable
- $p_j : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}, j = 1, 2, \ldots, l$ be polynomials

General Chance Constrained Optimization Problem

Solve

$$P_1^* = \max_{x} \text{Prob}_{\mu_q} \{ q : p_j(x, q) \geq 0, j = 1, 2, \ldots, l \}$$

In general, chance constrained problems are non convex.
Let

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**General Chance Constrained Optimization Problem**

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1. $q \in \mathbb{R}^m$ be a random variable with probability measure $\mu_q$
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**Chance Constrained Optimization**

Solve

$$P_1^* = \max_x \text{Prob}_{\mu_q} \{ q : p_j(x, q) \geq 0, j = 1, 2, \ldots, l \}$$

In general, chance constrained problems are non convex.
This class of problems is quite large and encompasses many problems.

- Controller design
  - Probabilistic robust controller design
  - Model predictive controller design in presence of random disturbances

- System identification

- Risk minimization and trust design in the area of economy and finance
The main idea is to find a tractable approximation for chance constraints.

- Scenario Approach
- Worst-Case Scenario
- Robust Solutions of Uncertain Linear Programs
- Upper Bound Approximation of Probabilities
- Bernstein Approximation
- Many more....
Our Contribution

Based on *measure* and *moment theory*, a *sequence of semidefinite problems* is obtained whose solution converges to the solution of chance constrained problem.

Objective

- Looking for *measure* with support on decision variable $x$ that satisfies the criteria of chance constraint problem
- Looking for *moments* of desired *measure*
- Semidefinite Relaxations
Based on *measure* and *moment theory*, a *sequence of semidefinite problems* is obtained whose solution converges to the solution of chance constrained problem.

**Objective**

- Looking for *measure* with support on decision variable $x$ that satisfies the criteria of chance constraint problem
- Looking for *moments* of desired *measure*
- Semidefinite Relaxations
We aim at finding a *measure* that satisfies certain criteria.

Assume

- Both $x$ and $q$ are bounded
- Without loss of generality, $x$ belongs to the hyper-cube $\chi = [-1, 1]^n$
- Probability measure $\mu_q$ satisfies $\text{supp}(\mu_q) \subseteq Q = [-1, 1]^m$

Consider a compact algebraic set:

$$\mathcal{K} = \{ (x, q) : p_j(x, q) \geq 0, j = 1, 2, \ldots, l \} \subseteq (\chi \times Q)$$

Equivalent Problem in Measure Space

Solve

$$P^*_2 = \max_{\mu, \mu_x} \int d\mu$$

subject to

$$\mu \preceq \mu_x \times \mu_q$$

$$\text{supp}(\mu_x) \subseteq \chi, \text{supp} (\mu) \subseteq \mathcal{K}$$
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\[ \text{Prob}_{\mu_q}\{ q : p_j(x, q) \geq 0, \ j = 1, 2, \ldots, l \} = \int_{K} d(\mu_x \times \mu_q) \]

\[ = \max_{\mu} \int d\mu : \mu \preceq \mu_x \times \mu_q; \text{supp}(\mu) \subseteq K \]
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Theorem 1: Problem 2 is equivalent to Problem 1 in the following sense:

- The optimal values are the same.
- If $\mu_x^*$ be a solution of Problem 2, then, any $x^* \in supp(\mu_x^*)$ is a solution of Problem 1.
- If $x^*$ be a solution of Problem 1, then $\mu_x^* = \delta_{x^*}$ is a solution of Problem 2.
An Equivalent Problem in the Moment Space

We aim at finding a sequence of moments of a measure that satisfies the criteria of Problem 2.

**Preliminary Results**

- The sequence $\mathbf{y} = (y_\alpha)$ has a representing finite Borel measure $\mu$ on compact set
  $$K = \{ x \in \mathbb{R}^n : p_j(x) \geq 0, \ j = 1, 2, \ldots, m \}$$
  for some polynomials $p_j \in \mathbb{R}[x]$. If
  $$M_d(\mathbf{y}) \succeq 0, \ M_d( p_j \mathbf{y}) \succeq 0, \ j = 1, \ldots, m$$
  for every $d \in \mathbb{N}^n$.

- Given two measures $\mu_1$ and $\mu_2$ on a compact set $K$, with moment sequences $\mathbf{y}_1 = (y_{1\alpha})$ and $\mathbf{y}_2 = (y_{2\alpha})$, we have $\mu_1 \preceq \mu_2$ if :
  $$M_d(\mathbf{y}_2 - \mathbf{y}_1) \succeq 0, \ M_d( p_j(\mathbf{y}_2-\mathbf{y}_1)) \succeq 0, \ j = 1, \ldots, m$$
  for every $d \in \mathbb{N}^n$. 
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- Given two measures \( \mu_1 \) and \( \mu_2 \) on a compact set \( K \), with moment sequences \( y_1 = (y_1\alpha) \) and \( y_2 = (y_2\alpha) \), we have \( \mu_1 \preceq \mu_2 \) if :

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M_d(y_2 - y_1) \succeq 0, \ M_d(p_j(y_2 - y_1)) \succeq 0, \ j = 1, \ldots, m
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An Equivalent Problem in the Moment Space

Example of moment matrix

\[
M_N(m) = \begin{bmatrix}
M_{0,0}(m) & M_{0,1}(m) & \cdots & M_{0,N}(m) \\
M_{1,0}(m) & M_{1,1}(m) & \cdots & M_{1,N}(m) \\
\vdots & \vdots & \ddots & \vdots \\
M_{N,0}(m) & M_{N,1}(m) & \cdots & M_{N,N}(m)
\end{bmatrix}
\]

\[
M_{j,k}(m) = \begin{bmatrix}
m_{j+k,0} & m_{j+k-1,1} & \cdots & m_{j,k} \\
m_{j+k-1,1} & m_{j+k-2,2} & \cdots & m_{j-1,k+1} \\
\vdots & \vdots & \ddots & \vdots \\
m_{k,j} & m_{k-1,j+1} & \cdots & m_{0,j+k}
\end{bmatrix}
\]

Localizing matrix \(M_{N_i}(p_i, m)\) is defined as

\[
M_{N_i}(p_i, m)(i, j) = \sum_{\alpha} p_{i,\alpha} m(\beta(i, j) + \alpha)
\]
An Equivalent Problem in the Moment Space

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\vdots & \vdots & \ddots & \vdots \\
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\end{bmatrix} \]

\[
M_{j,k}(m) = \begin{bmatrix}
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\vdots & \vdots & \ddots & \vdots \\
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An Equivalent Problem in the Moment Space

Let

- \( y \), \( y_x \), and \( \hat{y} \) be the infinite sequence of all moments of measures \( \mu \), \( \mu_x \), and \( \hat{\mu} = \mu_x \times \mu_q \), respectively.

An Equivalent Problem in Moment Space

Solve

\[
P_3^* = \sup_{y, y_x} y_0
\]

subject to

\[
M_\infty(y) \succeq 0
\]
\[
M_\infty(p_j y) \succeq 0, \quad j = 1, 2, \ldots, l
\]
\[
M_\infty(y_x) \succeq 0
\]
\[
M_\infty(\hat{y} - y) \succeq 0
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An Equivalent Problem in the Moment Space

Let

- \( y, y_x, \) and \( \hat{y} \) be the infinite sequence of all moments of measures \( \mu, \mu_x, \) and \( \hat{\mu} = \mu_x \times \mu_q \), respectively.

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Chance Constrained Algebraic Problems
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Semidefinite Relaxation

Let
\[ y = (y_\alpha), \quad y_x = (y_{x\alpha}), \quad \text{and} \quad \hat{y} = (\hat{y}_\alpha) \]
be a sequence with appropriate dimension.

Semidefinite Relaxation

Solve

\[ P_{4i}^* = \sup_{y, y_x} y_0 \]

subject to

\[ M_i(y) \succeq 0 \]
\[ M_{i-j} (p_j y) \succeq 0, \quad j = 1, 2, \ldots, l \]
\[ M_i(y_x) \succeq 0 \]
\[ M_i(\hat{y} - y) \succeq 0 \]

Theorem 2

**Theorem 2:** Optimal value of problem \( P_{4i} \) converges to optimal value of problem \( P_3 \) as \( i \to \infty \).
Semidefinite Relaxation

Let
- $\mathbf{y} = (y_\alpha)$, $\mathbf{y}_x = (y_{x\alpha})$, and $\hat{\mathbf{y}} = (\hat{y}_\alpha)$ be a sequence with appropriate dimension.

Semidefinite Relaxation

Solve

$$P_{4}^* = \sup_{\mathbf{y}, \mathbf{y}_x} y_0$$

subject to

$$M_i(\mathbf{y}) \succeq 0$$
$$M_{i-r_j}(p_j \mathbf{y}) \succeq 0, \ j = 1, 2, \ldots, l$$
$$M_i(\mathbf{y}_x) \succeq 0$$
$$M_i(\hat{\mathbf{y}} - \mathbf{y}) \succeq 0$$

Theorem 2

Theorem 2: Optimal value of problem $P_{4}^i$ converges to optimal value of problem $P_3$ as $i \to \infty$. 
The optimum is achieved by a distribution $\mu_x$ whose support is a single point. Such distributions, have a moment matrix which has rank one.

### Rank Minimization

Solve

$$\min \| M_i(y_x) \|_*$$

subject to

$$y_0 \geq \gamma$$

$$M_i(y_x) \succeq 0$$

$$M_i(y) \succeq 0$$

$$M_{i-r_j}(p_jy) \succeq 0, \ j = 1, 2, \ldots, l$$

$$M_i(\hat{y} - y) \succeq 0$$

where $\| \cdot \|_*$ stands for nuclear norm, $i$ is larger than a maximum degree of polynomials $p_j(x, q)$ and $0 \leq \gamma \leq 1$. This is coupled with a line search on $\gamma$ to maximize the value of $y_0$. 
Implementation

The optimum is achieved by a distribution $\mu_x$ whose support is a single point. Such distributions, have a moment matrix which has rank one.

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Numerical Example 1

Consider chance constraint problem

$$\max_x \text{Prob}_{\mu_q} \left\{ q : p(x, q) = -\frac{1}{2} q (q^2 + (x - \frac{1}{2})^2) + (q^4 + q^2 (x - \frac{1}{2})^2 + (x - \frac{1}{2})^4) \geq 0 \right\}$$

The uncertain parameter $q : \mu_q = U[-1, 1]$

Figure: The semialgebraic set $p(x, q)$ of example 1
Numerical Example 1

Moment Vectors

Moment vector of measure $\mu$

$y = [y_{00}, y_{01}, y_{02}, y_{10}, y_{11}, y_{12}, y_{20}, y_{21}, y_{22}, y_{30}, y_{31}, y_{32}, y_{40}, y_{41}, y_{42}, y_{43}, y_{44}]$

Moment vector of measure $\mu_x$

$y_x = [1, y_{x1}, y_{x2}, y_{x3}, y_{x4}]$

Moment vector of measure $\mu_q$

$y_q = [1, y_{q1}, y_{q2}, y_{q3}, y_{q4}] = [1, 0, \frac{1}{3}, 0, \frac{1}{5}, 0]$

Moment vector of measure $\hat{\mu} = \mu_x \times \mu_q$

$y_x y_q = [1|y_{x1}, y_{q1}|y_{x2}, y_{x1}y_{q1}, y_{x3}, y_{x2}y_{q1}, y_{x1}y_{q2}, y_{x3}y_{q1}, y_{x2}y_{q2}, y_{x1}y_{q3}, y_{q4}]$

$= [1|y_{x1}, 0|y_{x2}, 0, \frac{1}{3}|y_{x3}, 0, \frac{1}{3}y_{x1}, 0|y_{x4}, 0, \frac{1}{3}y_{x2}, 0, \frac{1}{5}]$
Numerical Example 1

Semi Definite Program

\[ \min_{\gamma, y_{ij}, y_{xk}} \| M_4(yx) \| = \| \begin{pmatrix} 1 & y_{x1} & y_{x2} \\ y_{x1} & y_{x2} & y_{x3} \\ y_{x2} & y_{x3} & y_{x4} \end{pmatrix} \| \]

Subject to

\[ M_4(y) \succeq 0 \Rightarrow \left( \begin{array}{ccccccc} y_{00} & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} \\ y_{10} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\ y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\ y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22} \\ y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\ y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04} \end{array} \right) \succeq 0 \]

\[ M_4(yx yq) - M_4(y) \succeq 0 \Rightarrow \left( \begin{array}{ccccccc} 1 & y_{x1} & 0 & y_{x2} & 0 & 1/3 \\ y_{x1} & y_{x2} & 0 & y_{x3} & 0 & 1/3 y_{x1} \\ 0 & 0 & 1/3 & y_{x4} & 0 & 1/3 y_{x2} \\ 0 & 0 & 0 & 1/3 y_{x1} & 0 & 2/5 \\ 1/3 & 1/3 y_{x1} & 0 & 1/3 y_{x2} & 0 & \end{array} \right) \]

\[ M_4(py) \succeq 0 \Rightarrow -y_{04} + \frac{1}{2} y_{03} - y_{22} + y_{12} - \frac{1}{4} y_{02} + \frac{1}{2} y_{21} - \frac{1}{2} y_{11} + \frac{1}{8} y_{01} - y_{40} + 2 y_{30} - \frac{3}{2} y_{20} + \frac{1}{2} y_{10} - \frac{1}{16} \succeq 0 \]
Numerical Example 1

Results

Obtained Moments

\[ y = [0.58, 0.49, -0.15, 0, 0, 0.18, 0, 0, 0, -0.07, 0, 0, 0, 0] \]

Eigenvalues of \( M_4(x) \): \([0, 0, 0, 0.1, 1]\) \(: \text{Rank} (M_4(x)) \simeq 1 : \mu_x \simeq \text{Diracmeasure} \)

Optimal \( x^* \) : \( y_{x_1} = 0.499 \)
Optimal Probability : \( y_{00} = 0.58 \)
Consider chance constraint problem as:

\[
\max_x \text{Prob}_{\mu_q} \left\{ q : \left[ \begin{array}{ccc} q_1 & q_2 & q_3 \\ x_1 \\ x_2 \\ x_3 \end{array} \right] \geq 1, -1 \leq x_1, x_2, x_3 \leq 1 \right\}
\]

\(q_1, q_2, q_3\): independent uncertain parameters, \(\mu_{q_1}, \mu_{q_2}, \mu_{q_3} = U[-1, 1]\)

Obtained Optimal \(x^* = (0.94, 0.9, 0.9)\)
Obtained Optimal probability : \(y_{000} = 0.31\)

Monte Carlo method:
Obtained Optimal \(x^* = (1, 1, 1)\)
Obtained Optimal probability 0.17
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- A novel approach to approximately solving a large class of chance constrained problems
- A sequence of semidefinite relaxations whose solution converge to the optimum of the original problem

Further research

- Study of the convergence rate of the approximations
- Development of more efficient algorithms to solve each one of the relaxed problems.
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